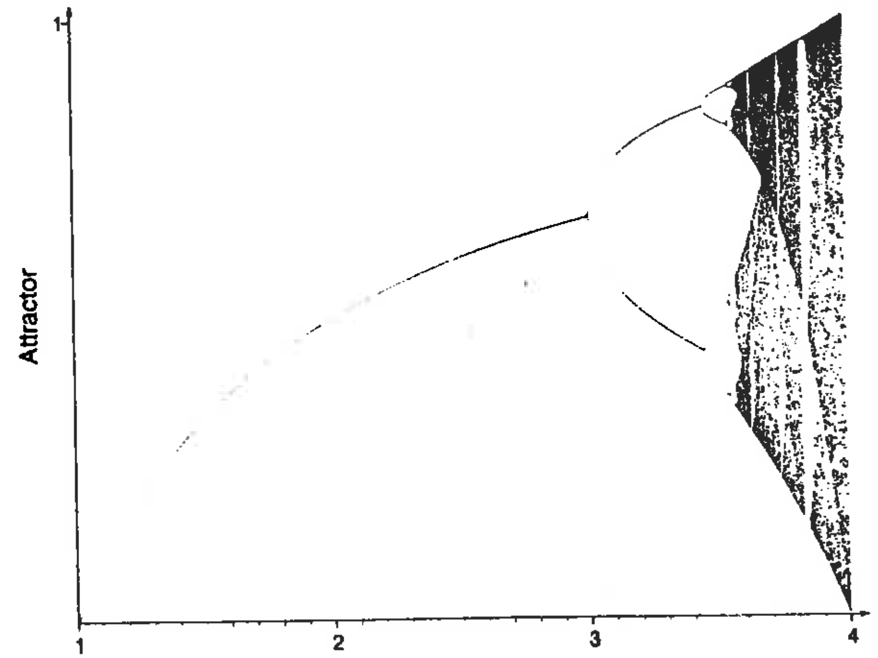
THE FEIGENBAUM PLOT

The quadratic function $f(x) = ax(1 - x)$ clearly exhibits an extraordinary variety of changing behaviors as the parameter a increases in value from 1 through 4. The most significant connection is with predictability. A stable, predictable long term behavior exists in the form of a fixed point attractor for low values of a. As a increases, predictability becomes more complex as the number of points in the attractors increases. As a approaches 4, the behavior moves quickly into an erratic, chaotic, unpredictable state.

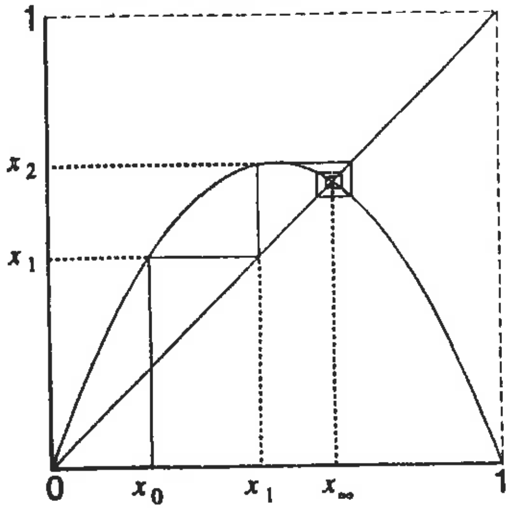
The consequence of this observation is profound and disturbing. Natural phenomena governed by such a quadratic relationship may appear to be in or out of control solely because of the functional relationship among the parameters and variables. Under one set of conditions they can be totally predictable yet, for only slight changes in conditions, they become completely chaotic and unpredictable.

A completed plot of attractors for all functions of the form $f(x) = ax(1 - x)$ for every parameter value from 1 through 4 is shown below. This complete plot is named after the American physicist Mitchell J. Feigenbaum. His work during the mid-1970's at Los Alamos Laboratory highlighted important properties associated with plots of this type.

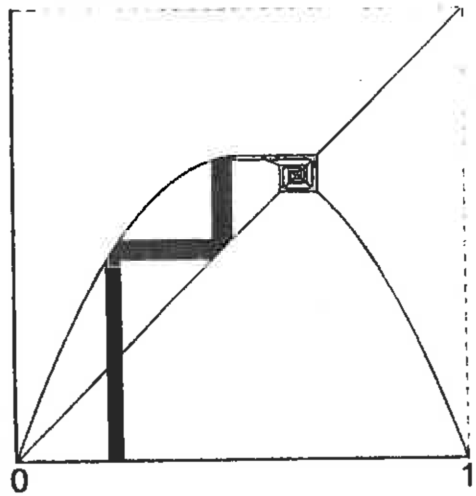


Attraction to a Point

Graphical iteration of an initial point leads to an attracting final state.

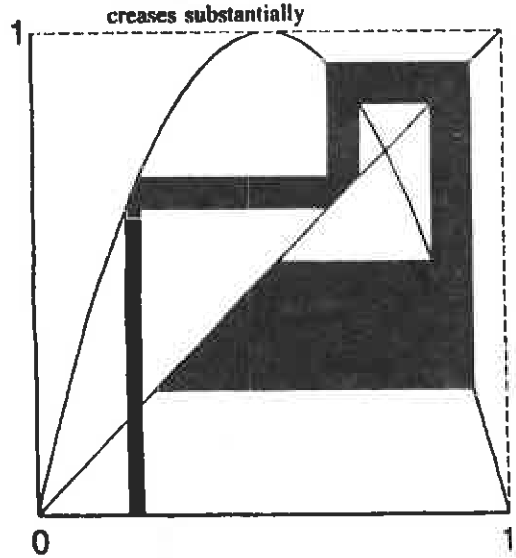


... the iteration of an entire interval of initial values is contracted into the final stable state.

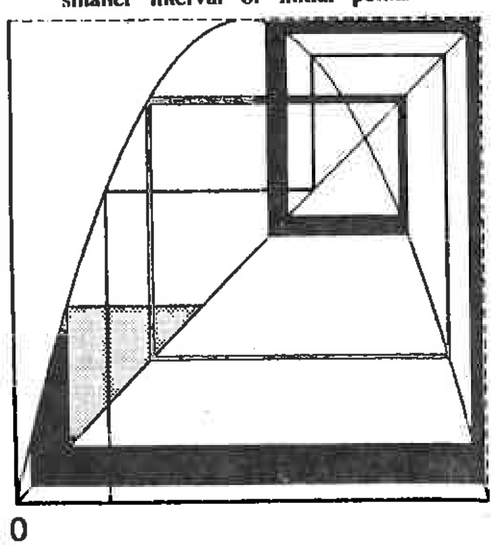


Sensitivity

Sensitivity demonstrated by graphical iteration: in the course of the iteration even a small deviation increases substantially



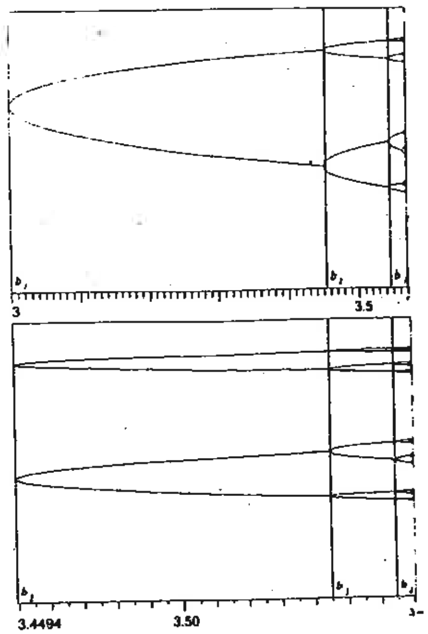
The experiment is repeated with an even smaller interval of initial points



Precisely at $a = b_1 = 3$ there is the transition of the attractor from a fixed point to a 2-cycle. This starts a whole sequence of so-called *period-doubling bifurcations.* At $a = b_2 = 3.4494...$ the 2-cycle changes into a 4-cycle, at $a = b_3 = 3.544...$ the 4-cycle becomes an 8-cycle, and so on. The upper figure on the right shows the Feigenbaum plot for the parameter range from the first bifurcation at $a = b_1 = 3$ to $a = 3.569945...$ which is approximately the point where the sequence of period-doublings ends. We call this part of the plot the period-doubling tree. The lower figure is a close up that shows the part of the period-doubling tree starting at $a = 3.4494...$ The location of the bifurcation points in both figures match almost exactly on the parameter axes. In terms of a formula this means

$$\frac{b_2 - b_1}{b_3 - b_2} \approx \frac{b_3 - b_2}{b_4 - b_3} \approx \frac{b_4 - b_3}{b_5 - b_4} \approx \cdots$$

Let us explore this property.



15. Here is a list of the first seven bifurcation points. Compute the differences.

$b_1 = 3.0$
$b_2 = 3.449490...$ $\quad d_1 = b_2 - b_1 =$ ____________
$b_3 = 3.544090...$ $\quad d_2 = b_3 - b_2 =$ ____________
$b_4 = 3.564407...$ $\quad d_3 = b_4 - b_3 =$ ____________
$b_5 = 3.568759...$ $\quad d_4 = b_5 - b_4 =$ ____________
$b_6 = 3.569692...$ $\quad d_5 = b_6 - b_5 =$ ____________
$b_7 = 3.569891...$ $\quad d_6 = b_7 - b_6 =$ ____________

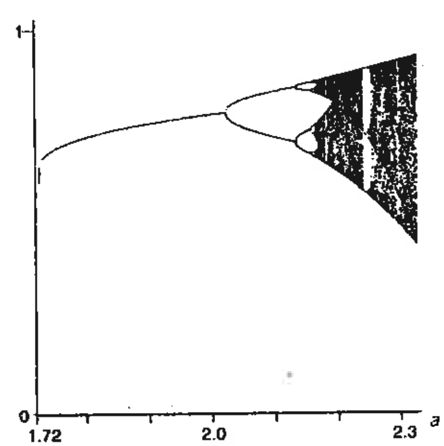
16. Based on these results compute the ratios of the successive differences d_k/d_{k+1}.

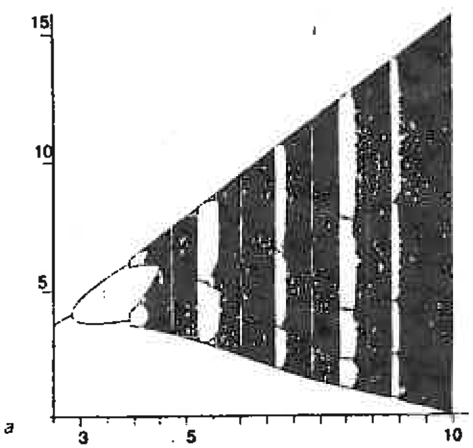
d_1/d_2	d_2/d_3	d_3/d_4	d_4/d_5	d_5/d_6

Note, that these ratios from bifurcation to bifurcation are not exactly identical but seem to tend to a certain value. In fact, this is one of the great discoveries of Feigenbaum. The ratios converge with increasing k to 4.669202..., which is called the *Feigenbaum number.*

$$\lim_{k\to\infty} \frac{d_k}{d_{k+1}} = \delta = 4.669202...$$

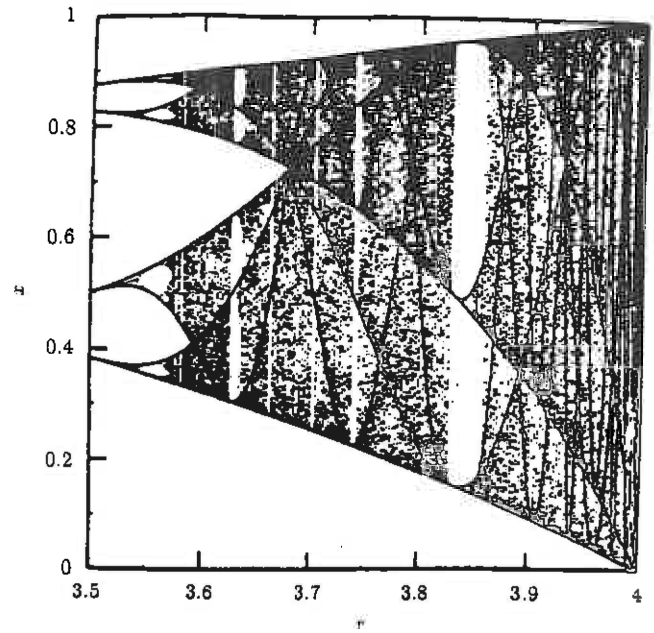
At first, it seems that this is just another number which documents the behavior of our particular example, the quadratic iteration. However, Feigenbaum went on to demonstrate that the character of δ is quite different. The number $\delta = 4.669202...$ is ***universal.*** It is the same for a wide range of different iterators. For example, it occurs in the logistic equation (see Activity 5.8) or the iteration of the trigonometric function $g_a(x) = ax^2 \sin(\pi x)$ with parameter a.





The constant δ can also be found in the study of certain differential equations and even in physical experiments ranging from turbulent flow to electronic circuits. The left figure shows the Feigenbaum plot for $g_a(x)$. It looks very similar to the plot of the quadratic iterator. The right figure shows the plot for a set of differential equations, the so called Rössler system.

In all these cases we can observe the period-doubling bifurcation and the analysis of the sequence of bifurcation points leads to the universal Feigenbaum constant δ.



The different period lengths p of stable periodic orbits of unimodal maps appear in a *universal order*. If r_p is the value of the growth parameter r at which a stable period of length p first appears as r is increased, then $r_p > r_q$ if $p \succ q$ (read p precedes q) in the following "Sharkovskii order":

$$3 \succ 5 \succ 7 \succ 9 \succ \cdots$$
$$2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots$$
$$\cdots$$
$$2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \cdots$$
$$\cdots$$
$$\cdots \succ 2^m \succ \cdots \succ 4 \succ 2 \succ 1$$

Thus, for example, the minimal r value for an orbit with $p = 10 = 2 \cdot 5$ is larger than the minimal r value for $p = 12 = 4 \cdot 3$ because $10 \succ 12$ in this witchcraft algebra.

Some of the consequences of this ordering are the following:

- The existence of period length $p = 3$ guarantees the existence of any other period length q for some $r_q < r_p$.
- If only a finite number of period lengths occur, their lengths must be powers of 2—that is, $p = 2^k, 2^{k-1}, \ldots, 4, 2, 1$, for some k.
- If a period length p exists that is not a power of 2, then there are infinitely many different periods.